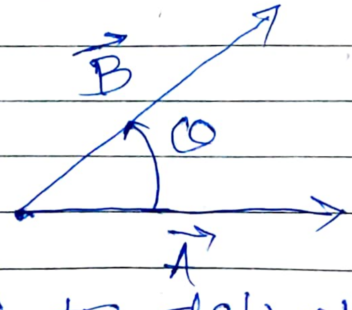
Vector Algebra

Vector product

The product of two directed quantities (vectors) may be result into a directed quantity (vector) or undirected quantity (scalar).

Hence there will be two different product operations for vectors. One resulting to a scalar and another resulting to a vector. These are known as dot product product and cross product respectively.

The scalar product of two vectors $\vec{A}$ and $\vec{B}$ represented by the symbol $\vec{A} \cdot \vec{B}$ and is read as vector $\vec{A}$ dot vector $\vec{B}$ for this reason the scalar product is also called as dot product. The scalar product or dot product of two vectors $\vec{A}$ and $\vec{B}$ is defined as a scalar whose value is equal to product of magnitudes of two vectors and cosine of the angle between them



According to defination we can write

$$\vec{A} \cdot \vec{B} = AB\cos\theta.$$

If both vectors are perpendicular to each other their scalar or dot product is zero.

i.e $\vec{A}$ is perpendicular to $\vec{B}$

then

$$\vec{A}\cdot\vec{B} = AB\cos 90^\circ$$
$$= AB(0) \qquad \because \cos 90^\circ = 0$$
$$= 0$$

The condition of perpendicularity is expressed by

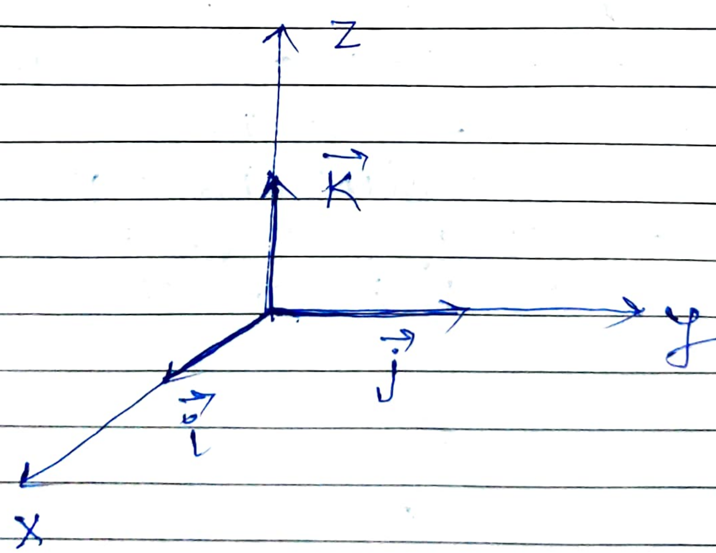
$$\vec{A}\cdot\vec{B} = 0$$

clearly

$$\vec{A}\cdot\vec{A} = AA\cos 0^\circ$$
$$= AA1$$
$$= A^2 \qquad \because \cos 0^\circ = 1$$

since the angle between them is zero.

The scalar or dot product among the unit vectors $\vec{i}$, $\vec{j}$ & $\vec{K}$ along the three co-ordinate axes



$$\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{K}\cdot\vec{K} = 1$$
$$\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{K} = \vec{K}\cdot\vec{i} = 0$$
$$\vec{i}\cdot\vec{K} = \vec{K}\cdot\vec{j} = \vec{j}\cdot\vec{i} = 0$$

The above relation can be obtained easily by the use of defination of dot product and noting that angle

* between two equal unit vectors is zero where as angle between two different cartesian axes unit vectors is 90° (degree)

The dot or scalar product of two vectors can be expressed interms of cartesian components of the two vectors
If A_x, A_y, A_z and B_x, B_y, B_z are the components of vectors $\vec{A}$ and $\vec{B}$ then

$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{K}$$

$$\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{K}$$

$$(\vec{A}\cdot\vec{B}) = (A_x\vec{i} + A_y\vec{j} + A_z\vec{K})\cdot(B_x\vec{i} + B_y\vec{j} + B_z\vec{K})$$

Apply the distributive law with respective to their sum

$$= A_xB_x(\vec{i}\cdot\vec{i}) + A_xB_y(\vec{i}\cdot\vec{j}) + A_xB_z(\vec{i}\cdot\vec{K}) + A_yB_x(\vec{j}\cdot\vec{i}) + A_yB_y(\vec{j}\cdot\vec{j}) + A_yB_z(\vec{j}\cdot\vec{K}) + A_zB_x(\vec{K}\cdot\vec{i}) + A_zB_y(\vec{K}\cdot\vec{j}) + A_zB_z(\vec{K}\cdot\vec{K})$$

Apply the law of scalar product among the unit vectors $\vec{i}$, $\vec{j}$ and $\vec{K}$

$$= A_xB_x + 0 + 0 + 0 + A_yB_y + 0 + 0 + 0 + A_zB_z$$

$$= A_xB_x + A_yB_y + A_zB_z$$

$$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

Ex-

Find the vector product of two vectors

$\vec{A} = 2\vec{i} - 3\vec{j} + \vec{K}$

$\vec{B} = 4\vec{i} + \vec{j} - 5\vec{K}$

$\vec{A} \cdot \vec{B} = (2\vec{i} - 3\vec{j} + \vec{K}) \cdot (4\vec{i} + \vec{j} - 5\vec{K})$

Apply distributive law

$= (2)(4)(\vec{i} \cdot \vec{i}) + 2(\vec{i} \cdot \vec{j}) - 10(\vec{i} \cdot \vec{K})$

$- (3)(4)(\vec{j} \cdot \vec{i}) - 3(\vec{j} \cdot \vec{j}) + 15(\vec{j} \cdot \vec{K})$

$+ 4(\vec{K} \cdot \vec{i}) + (\vec{K} \cdot \vec{j}) - 5(\vec{K} \cdot \vec{K})$

$= 8 + 0 - 0 - 0 - 3 + 0 + 0 + 0 - 5$

$= 8 - 3 - 5$

$= 8 - 8$

$= 0$

$\vec{A} \cdot \vec{B} = 0$

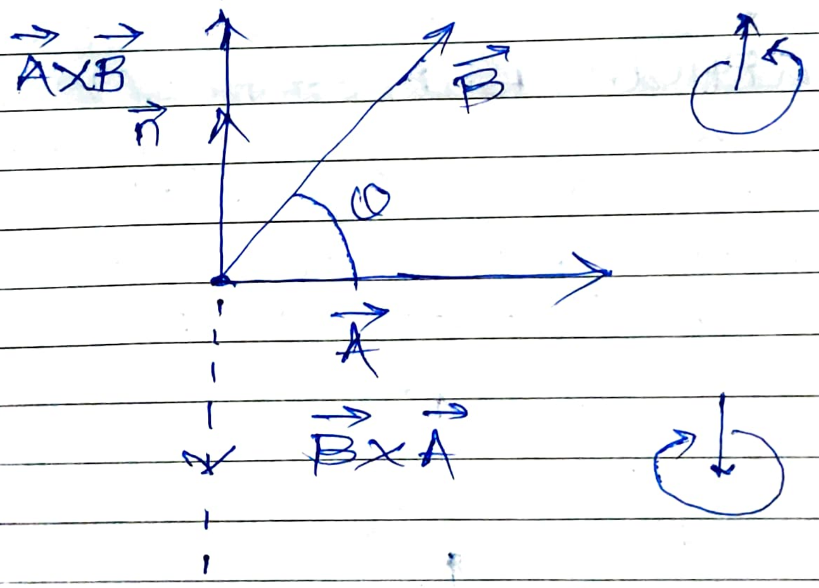
As the scalar or dot product is zero the vector must be perpendi-cular to each other.

~~cross of vector~~

Cross-product

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is defened as a vector whose value is equal to product of magnitude of two vectors and sine of the angle between them and direction is perpendicular to the plane of $\vec{A}$ and $\vec{B}$ and is decided by right handed screw rule.

Cross or vector product is represented by $\vec{A} \times \vec{B}$ and read as vector $\vec{A}$ cross vector $\vec{B}$



According to defination

$$\vec{A}\times\vec{B} = AB\sin\theta\,\vec{n}$$

If the right handed screw is rotated from $\vec{A}$ to $\vec{B}$ through a small angle then direction of advancement of tip of screw is the direction of $\vec{n}$

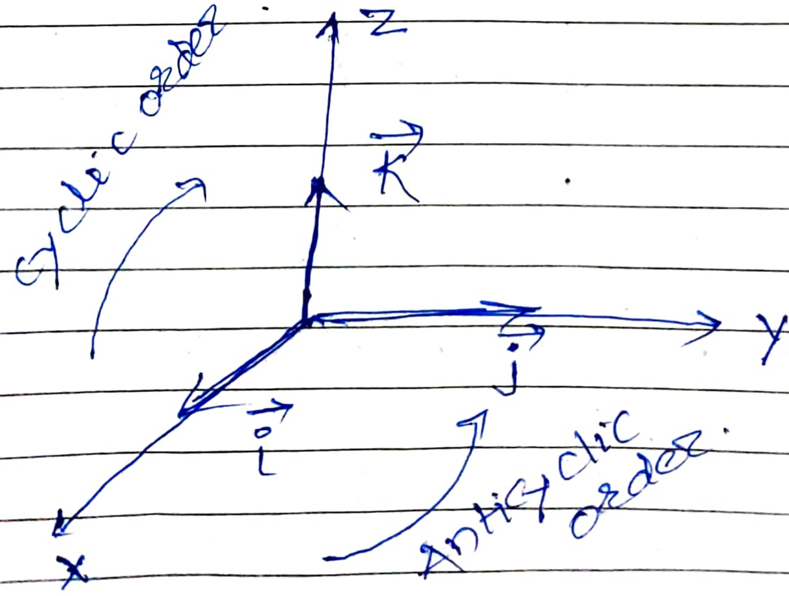
cross product of two non zero vectors are zero if the angle between them is zero degree or 180 degree and sin θ is zero. Hence if the cross product of two vectors is zero then the two vectors are either parallel or antiparallel. The zero value of cross product is useful in testing co-linearity of two vectors

The cross product obeys the following laws

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$\vec{A} \times \vec{B}$ and $\vec{B} \times \vec{A}$ have the same magnitude but opposite direction as shown in the figure. So the cross product is anticommutative

The cross product among the three unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ are



$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$$

$\vec{i}\times\vec{j}=K,\ \vec{j}\times\vec{K}=\vec{i},\ \vec{K}\times\vec{i}=\vec{j}$

but

$\vec{i}\times\vec{K}=-\vec{j},\ \vec{K}\times\vec{j}=-\vec{i},\ \vec{j}\times\vec{i}=-\vec{K}$

Cross or vector product is distributive relative to their sum, that is

$$\vec{A}\times(\vec{B}+\vec{C})=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

* The cross product of two vectors can be expressed interms of its components in the following manner

If A_x, A_y, A_z and B_x, B_y, B_z are the components of vectors $\vec{A}$ and $\vec{B}$

$$\vec{A}=A_x\vec{i}+A_y\vec{j}+A_z\vec{K}$$

$$\vec{B}=B_x\vec{i}+B_y\vec{j}+B_z\vec{K}$$ then

$$\vec{A}\times\vec{B}=(A_x\vec{i}+A_y\vec{j}+A_z\vec{K})\times(B_x\vec{i}+B_y\vec{j}+B_z\vec{K})$$

Apply the distributive law.

$$=A_xB_x(\vec{i}\times\vec{i})+A_xB_y(\vec{i}\times\vec{j})+A_xB_z(\vec{i}\times\vec{K})$$

$$+A_yB_x(\vec{j}\times\vec{i})+A_yB_y(\vec{j}\times\vec{j})+A_yB_z(\vec{j}\times\vec{K})$$

$$+A_zB_x(\vec{K}\times\vec{i})+A_zB_y(\vec{K}\times\vec{j})+A_zB_z(\vec{K}\times\vec{K})$$

by using the property of vector product among the unit vectors $\vec{i}, \vec{j}$ & $\vec{K}$ we get

$\vec{A} \times \vec{B}$

$= \cancel{A_x B_x \cdot 0 + A_y B_y}$

$= A_x B_x \cdot 0 + A_x B_y \vec{k} + A_x B_z (-\vec{j})$

$+ A_y B_x (-\vec{k}) + A_y B_y \cdot 0 + A_y B_z (\vec{i})$

$+ A_z B_x (\vec{j}) + A_z B_y (-\vec{i}) + A_z B_z \cdot 0$

$= (A_y B_z - A_z B_y) \vec{i} + (A_z B_x - A_x B_z) \vec{j}$

$+ (A_x B_y - A_y B_x) \vec{k}$

Above equation may be written in the more compact determinant form.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

is the representation of vector or cross product of two vectors as a determinant.

Find the vector product of two vectors

i) $\vec{A} = 2\vec{i} + 3\vec{j} + 5\vec{k}$, $\vec{B} = 3\vec{i} - 4\vec{j} + 6\vec{k}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 5 \\ 3 & -4 & 6 \end{vmatrix}$$

$$= \vec{i}[18-(-20)] + \vec{j}[15-12] + \vec{k}[-8-9]$$

$$= 38\vec{i} + 3\vec{j} - 17\vec{k}.$$

ii) Show that the vectors $\vec{A} = 2\vec{i} - 3\vec{j} - \vec{k}$, $\vec{B} = -6\vec{i} + 9\vec{j} + 3\vec{k}$ are parallel.

The two vectors will be parallel if their vector product is zero.

$$\vec{A}\times\vec{B} = 0$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & -1 \\ -6 & 9 & 3 \end{vmatrix}$$

$$= \vec{i}[-9-(-9)] + \vec{j}[6-6] + \vec{k}[18-18]$$

$$= \vec{i}[-9+9] + \vec{j}[6-6] + [18-18]\vec{k}$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$= 0$$

Hence two vectors are parallel to each other.

3) $\vec{A} = 2\vec{i} + \vec{j} - \vec{k}$

$\vec{B} = 3\vec{i} - 2\vec{j} + 4\vec{k}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 3 & -2 & 4 \end{vmatrix}$$

$$= \vec{i}\,[4-2] + \vec{j}\,[-3-8] + \vec{k}\,[-4-3]$$

$$= 2\vec{i} - 11\vec{j} - 7\vec{k}$$

Triple product of vectors

We stated that the vector product of two vectors $\vec{B}$ and $\vec{C}$ is a vector quantity. So this product $(\vec{B}\times\vec{C})$ may be multiplied scalarly or vectorially with a third vector $\vec{A}$ to give two triple product namely $\vec{A}\cdot(\vec{B}\times\vec{C})$ and $\vec{A}\times(\vec{B}\times\vec{C})$. The former being scalar quantity is termed as scalar triple product and latter being vector quantity is called vector triple product.

Scalar Triple Product

Def^n. Let $\vec{A}, \vec{B}, \vec{C}$ be the three vectors. then scalar product of any of these vector with vector product of other two such as $\vec{A}\cdot(\vec{B}\times\vec{C})$ is called scalar triple product of the vectors $\vec{A}, \vec{B}, \vec{C}$ and is denoted by $[\vec{A}\vec{B}\vec{C}]$ or $[\vec{A},\vec{B},\vec{C}]$ obviously this type of triple product is a scalar quantity. This scalar triple product is some times known as box product.

The expression for scalar triple product interms of components of the vectors can be found as follows

$$[\vec{A}\vec{B}\vec{C}] = \vec{A}\cdot(\vec{B}\times\vec{C})$$

$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{K}$$
$$\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{K}$$
$$\vec{C} = C_x\vec{i} + C_y\vec{j} + C_z\vec{K}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (A_x\vec{i} + A_y\vec{j} + A_z\vec{K})\cdot\begin{vmatrix}\vec{i} & \vec{j} & \vec{K}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

$$= (A_x\vec{i} + A_y\vec{j} + A_z\vec{K})\cdot\left[\vec{i}(B_yC_z - B_zC_y) + \vec{j}(B_zC_x - B_xC_z) + \vec{K}(B_xC_y - B_yC_x)\right]$$

$$= (\vec{i}\cdot\vec{i})A_x[B_yC_z - B_zC_y] + (\vec{i}\cdot\vec{j})A_x[B_zC_x - B_xC_z] + (\vec{i}\cdot\vec{K})A_x[B_xC_y - B_yC_x]$$
$$+ (\vec{j}\cdot\vec{i})A_y[B_yC_z - B_zC_y] + (\vec{j}\cdot\vec{j})A_y[B_zC_x - B_xC_z] + (\vec{j}\cdot\vec{K})A_y[B_xC_y - B_yC_x]$$
$$+ (\vec{K}\cdot\vec{i})A_z[B_yC_z - B_zC_y] + (\vec{K}\cdot\vec{j})A_z[B_zC_x - B_xC_z] + (\vec{K}\cdot\vec{K})A_z[B_xC_y - B_yC_x]$$

By using scalar product among unit vectors

$$\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{K}\cdot\vec{K} = 1$$
$$\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{K} = \vec{K}\cdot\vec{i} = 0$$

$$A_x[B_yC_z - B_zC_y] + 0 + 0 + 0 + A_y[B_zC_x - B_xC_z] + 0 + 0 + 0 + A_z[B_xC_y - B_yC_x]$$

$$A_x[B_yC_z - B_zC_y] + A_y[B_zC_x - B_xC_z] + A_z[B_xC_y - B_yC_x]$$

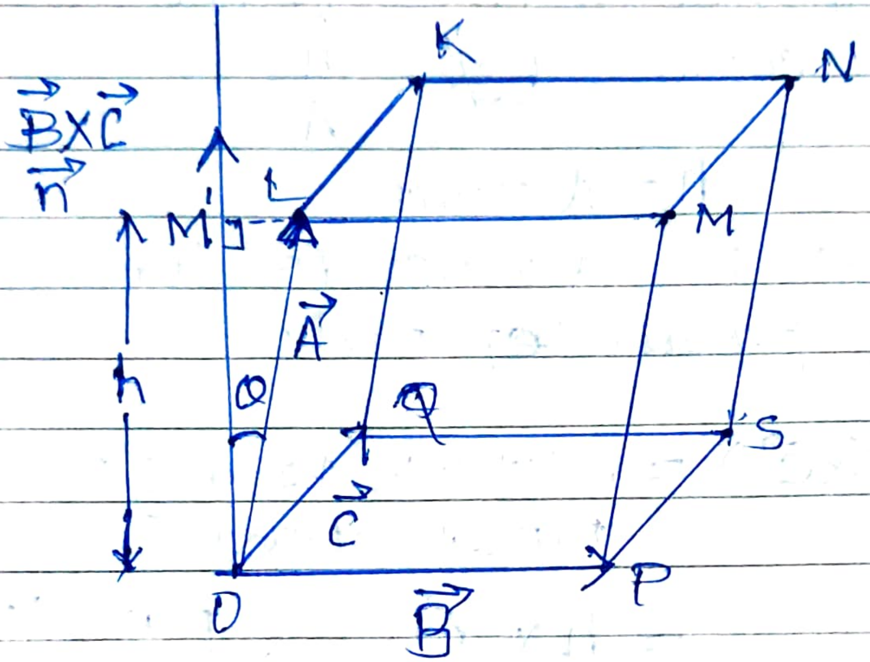
Page No. :
Date : / /

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= [\vec{A}\,\vec{B}\,\vec{C}]$$

Geometrical Interpretation of Scalar triple product

7



Three non zero vectors $\vec{A}$, $\vec{B}$ & $\vec{C}$ are represented by three adjacent sides OL, OP and OQ of a parallelepiped. Vector $\vec{B}\times\vec{C}$ is perpendicular to the plane of $\vec{B}$ & $\vec{C}$.
The two vectors $\vec{B}$ and $\vec{C}$ represented by the sides of OP and OQ of parallelogram base OPSQ as shown in figure.

$(\vec{B}\times\vec{C})$ = (Area of parallelogram OPSQ formed by adjacent sides OP and OQ) $\vec{n}$

Let θ be the angle between $\vec{A}$ and $\vec{B}\times\vec{C}$

$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{A}\cdot\hat{n} \times$ Area of parallelogram OPSQ

but from the figure

$$\cos\theta = \frac{OM'}{OL} = \frac{h}{A}$$

$$h = A\cos\theta$$

where $h = A\cos\theta$ is the hight of parallelopiped.

$\vec{A}\cdot(\vec{B}\times\vec{C}) = A\cos\theta \times$ Area of parallelepiped OPSQ

$= h \times$ Area of parallelogram OPSQ

$\vec{A}\cdot(\vec{B}\times\vec{C}) =$ Volume of parallelepiped

Thus, the scalar triple product represents the volume of the parallelopiped fromed by three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ as the three adjacent side of it. Any face of this parallelopiped can be taken as the base.

Vector Triple product

Let us consider the three vectors

$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$$
$$\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{k}$$
$$\vec{C} = C_x\vec{i} + C_y\vec{j} + C_z\vec{k}$$

$$\boxed{\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})}$$

$$(\vec{B}\times\vec{C}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \vec{i}(B_yC_z - B_zC_y) + \vec{j}(B_zC_x - B_xC_z) + \vec{k}(B_xC_y - B_yC_x)$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ (\vec{B}\times\vec{C})_x & (\vec{B}\times\vec{C})_y & (\vec{B}\times\vec{C})_z \end{vmatrix}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ (B_yC_z - B_zC_y) & (B_zC_x - B_xC_z) & (B_xC_y - B_yC_x) \end{vmatrix}$$

$$= \vec{i}\left[A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)\right]$$
$$+ \vec{j}\left[A_z(B_yC_z - B_zC_y) - A_x(B_xC_y - B_yC_x)\right]$$
$$+ \vec{k}\left[A_x(B_zC_x - B_xC_z) - A_y(B_yC_z - B_zC_y)\right]$$

its x components

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{i}\left[A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)\right]$$

$$= \vec{i}\left[A_yB_xC_y - A_yB_yC_x - A_zB_zC_x + A_zB_xC_z\right]$$

or $$= \vec{i}\left[A_yC_yB_x - A_yC_xB_y - A_zC_xB_z + A_zC_zB_x\right]$$

$$= \vec{i}\left[A_yC_yB_x + A_zC_zB_x - A_yC_xB_y - A_zC_xB_z\right]$$

Add and Substract $A_xB_xC_x$

$$= \vec{i}\left[A_yC_yB_x + A_zC_zB_x + A_xB_xC_x - A_yC_xB_y - A_zC_xB_z - A_xB_xC_x\right]$$

$$= \vec{i}\left[B_x(A_yC_y + A_zC_z + A_xC_x) - C_x(A_yB_y + A_zB_z + A_xB_x)\right]$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{i}B_x(\vec{A}\cdot\vec{C}) - \vec{i}C_x(\vec{A}\cdot\vec{B}) \quad \text{--- (1)}$$

$$= \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \quad \text{--- (1)}$$

Its y components

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{j}\left[A_z(B_yC_z - B_zC_y) - A_x(B_xC_y - B_yC_x)\right]$$

$$= \vec{j}\left[A_zB_yC_z - A_zB_zC_y - A_xB_xC_y + A_xB_yC_x\right]$$

$$= \vec{j}\left[A_zB_yC_z + A_xB_yC_x - A_zB_zC_y - A_xB_xC_y\right]$$

Add & Substract $A_yB_yC_y$ ③ 9

$$= \vec{j}\,[A_zB_yC_z + A_xB_yC_x + A_yB_yC_y$$
$$- A_zB_zC_y - A_xB_xC_y - A_yB_yC_y]$$

$$= \vec{j}\,[B_y(A_zC_z + A_xC_x + A_yC_y)$$
$$- C_y(A_zB_z + A_xB_x + A_yB_y)]$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{j}B_y(\vec{A}\cdot\vec{C}) - \vec{j}C_y(\vec{A}\cdot\vec{B}) \quad \text{--- ②}$$

its z components

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{K}\,[A_x(B_zC_x - B_xC_z)$$
$$- A_y(B_yC_z - B_zC_y)]$$

$$= \vec{K}\,[A_xB_zC_x - A_xB_xC_z - A_yB_yC_z$$
$$+ A_yB_zC_y]$$

$$= \vec{K}\,[A_xB_zC_x + A_yB_zC_y - A_xB_xC_z$$
$$- A_yB_yC_z]$$

Add & Substract $A_zB_zC_z$

$$= \vec{K}\,[A_xB_zC_x + A_yB_zC_y + A_zB_zC_z$$
$$- A_xB_xC_z - A_yB_yC_z - A_zB_zC_z]$$

$$= \vec{K}\,[B_z(A_xC_x + A_yC_y + A_zB_z)$$
$$- C_z(A_xB_x + A_yB_y + A_zB_z)]$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{K}B_z(\vec{A}\cdot\vec{C}) - \vec{K}C_z(\vec{A}\cdot\vec{B}) \quad -③$$

Adding equation ①, ② & ③ we get.

$$\vec{A}\times(\vec{B}\times\vec{C}) = (B_x\vec{i} + B_y\vec{j} + B_z\vec{k})(\vec{A}\cdot\vec{C})$$
$$-(C_x\vec{i} + C_y\vec{j} + C_z\vec{k})(\vec{A}\cdot\vec{B})$$

$$\boxed{\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})} \quad -④$$

The divergence of Vector Field

If $\vec{V}(x,y,z) = V_x\vec{i} + V_y\vec{j} + V_z\vec{k}$ be a contineous differentiable vector point function specified in a vector field. Then the divergence of V is defined as

$$\vec{i}\cdot\frac{\partial V_x}{\partial x} + \vec{j}\cdot\frac{\partial V_y}{\partial y} + \vec{k}\cdot\frac{\partial V_z}{\partial z}$$

and it written as $\nabla\cdot\vec{V}$ or $\operatorname{div}\vec{V}$ and read as divergence $\vec{V}$.

$$\therefore \nabla\cdot\vec{V} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(V_x\vec{i} + V_y\vec{j} + V_z\vec{k}\right)$$

$$= (\vec{i}\cdot\vec{i})\frac{\partial V_x}{\partial x} + \vec{j}\cdot\vec{j}\frac{\partial V_y}{\partial y} + \vec{k}\cdot\vec{k}\frac{\partial V_z}{\partial z}$$

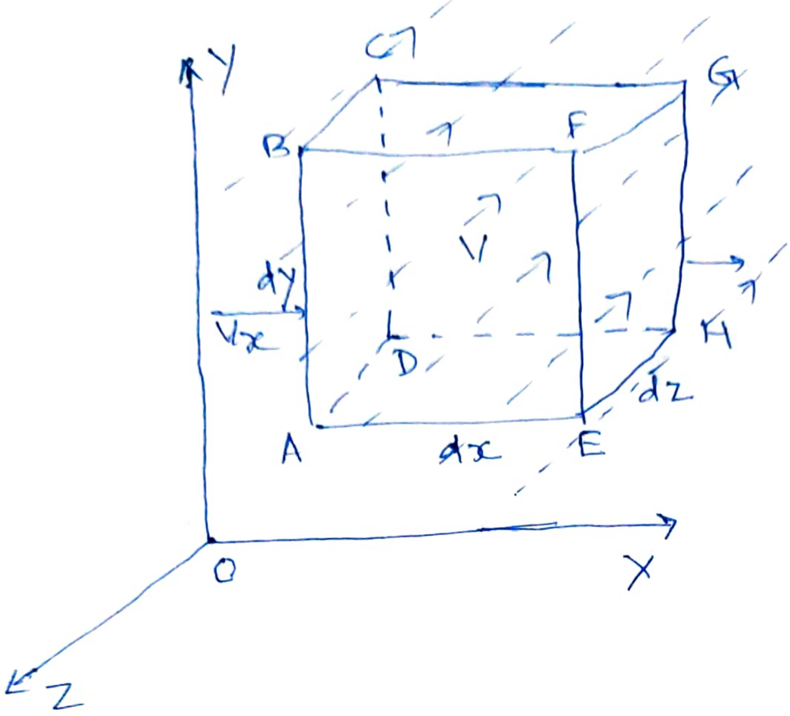
$$\nabla\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

which is scalar quantity

Physical Interpretation

The divergence of a vector field is a scalar quantity, and represents net amount of flow coming out per unit vol of a volume element of the fluid.

Now consider the flow of liquid with a velocity V at A and in the direction shown in Figure



Let V_x, V_y, V_z be the components of the velocity along x, y, z. Consider <u>infinitesimal</u> (small) volume element within the fluid with sides dx, dy, dz

If ρ be the mean density of the fluid

The volume of fluid entering the face ABCD

= density × Velocity normal to the face ABCD × Area of Face ABCD

$$= \rho V_x \, dy \, dz \quad \text{------} \quad (1)$$

Since velocity of Fluid coming out of the face EFGH $= V_x + \frac{\partial V_x}{\partial x} dx$

∴ Flow of the liquid coming out from EFGH

$$= \rho \left[V_x + \frac{\partial V_x}{\partial x} dx \right] dy \, dz \quad \text{(Diverging)} \quad (11)$$

rate of change of velocity (vector) per unit length × total length $\frac{\partial V_x}{\partial x} dx$

∴ net outward flow in the x direction from equation ① and ②

$$= \rho\left[V_x + \frac{\partial V_x}{\partial x}dx\right]dy\,dz - \rho V_x\,dy\,dz$$

$$= \cancel{\rho V_x\,dy\,dz} + \rho\frac{\partial V_x}{\partial x}dx\,dy\,dz - \cancel{\rho V_x\,dy\,dz}$$

$$= \rho\frac{\partial V_x}{\partial x}dx\,dy\,dz \quad \dots\dots \quad (3)$$

Simillarly the net outward flow leaving in the y and z directions

$$= \rho\frac{\partial V_y}{\partial y}dy\,dx\,dz \quad - y \text{ direction.}$$

$$= \rho\frac{\partial V_z}{\partial z}dz\,dy\,dx \quad - z \text{ direction}$$

Total net out-ward flow (or flux) per unit vol diverging from the element

$$= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

since the amount of flux per unit vol. is called divergence of V we write

$$\text{div } V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \text{ which is}$$

Scalar

simillarly if E is the electric intensity instead of V we get

$$\text{div } E = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Since $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ and

$$V = V_x\vec{i} + V_y\vec{j} + V_z\vec{k}$$

the scalar product of ∇ and V is given by

$$\nabla \cdot V = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(V_x\vec{i} + V_y\vec{j} + V_z\vec{k}\right)$$

$$= (\vec{i}\cdot\vec{i})\frac{\partial V_x}{\partial x} + (\vec{j}\cdot\vec{j})\frac{\partial V_y}{\partial y} + (\vec{k}\cdot\vec{k})\frac{\partial V_z}{\partial z}$$

$$\boxed{\nabla \cdot V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}}$$

Thus in general we can write for a vector function E as

$$\text{div } E = \nabla \cdot E = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

It will be notice from the above that by using ∇, the vector quantity has been converted into a scalar by using the dot product of ∇ and the vector.

If the div E = 0 the total flux entering the element dx dy dz equals that leaving the surface such a vector is called solenoidal or sourceless vector

The divergence of sum of two vector function

If U and V be two vector point functions expressed as $\nabla\cdot[(U_x\vec{i}+U_y\vec{j}+U_z\vec{k})+(V_x\vec{i}+V_y\vec{j}+V_z\vec{k})]$

$$U = U_x\vec{i}+U_y\vec{j}+U_z\vec{k}, \quad V = V_x\vec{i}+V_y\vec{j}+V_z\vec{k}$$

then

$$\nabla\cdot(U+V) = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot[(U_x+V_x)\vec{i} + (U_y+V_y)\vec{j}+(U_z+V_z)\vec{k}]$$

$$= \frac{\partial}{\partial x}(U_x+V_x)+\frac{\partial}{\partial y}(U_y+V_y)+\frac{\partial}{\partial z}(U_z+V_z)$$

$$= \frac{\partial U_x}{\partial x}+\frac{\partial V_x}{\partial x}+\frac{\partial U_y}{\partial y}+\frac{\partial V_y}{\partial y}+\frac{\partial U_z}{\partial z}+\frac{\partial V_z}{\partial z}$$

$$= \left(\frac{\partial U_x}{\partial x}+\frac{\partial U_y}{\partial y}+\frac{\partial U_z}{\partial z}\right)+\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right)$$

$$= \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot(U_x\vec{i}+U_y\vec{j}+U_z\vec{k}) + \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot(V_x\vec{i}+V_y\vec{j}+V_z\vec{k})$$

$$= \nabla\cdot U+\nabla\cdot V = \mathrm{div}\, U + \mathrm{div}\, V$$

showing that divergence of sum of two vector functions is equal to the sum of their divergences

* The divergence of product *

if $U = U_x\vec{i}+U_y\vec{j}+U_z\vec{k}$ is the vector point function, and V is scalar point function

then $$\nabla\cdot(UV) = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot[(U_x\vec{i}+U_y\vec{j}+U_z\vec{k})V]$$

$$= \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial y}\right)\cdot(VU_x\vec{i}+VU_y\vec{j}+VU_z\vec{k})$$

$$= \frac{\partial}{\partial x}(VU_x)+\frac{\partial}{\partial y}(VU_y)+\frac{\partial}{\partial z}(VU_z)$$

$$= u_x \frac{\partial v}{\partial x} + v\frac{\partial u_x}{\partial x} + u_y \frac{\partial v}{\partial y} + v \frac{\partial u_y}{\partial y}$$

$$+ u_z \frac{\partial v}{\partial z} + v \frac{\partial u_z}{\partial z}$$

$$= \left(u_x \frac{\partial v}{\partial x} + u_y \frac{\partial v}{\partial y} + u_z \frac{\partial v}{\partial z} \right)$$

$$+ v \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right)$$

$$= \left(\vec{i} \frac{\partial v}{\partial x} + \vec{j} \frac{\partial v}{\partial y} + \vec{k} \frac{\partial v}{\partial z} \right) \cdot \left(u_x \vec{i} + u_y \vec{j} + u_z \vec{k} \right)$$

$$+ v \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot \left(u_x \vec{i} + u_y \vec{j} + u_z \vec{k} \right)$$

$$= (\nabla v) \cdot u + v (\nabla \cdot u)$$

Ex — If $\underline{V} = x^2 z \vec{i} - 2y^3 z^2 \vec{j} + xy^2 z \vec{k}$

find $\nabla \cdot v$ at the point $(1, -1, 1)$

$$\nabla \cdot v = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot \left(x^2 z \vec{i} - 2y^3 z^2 \vec{j} + xy^2 z \vec{k} \right)$$

$$= \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-2y^3 z^2) + \frac{\partial}{\partial z}(xy^2 z)$$

$$= 2xz - 6y^2 z^2 + xy^2$$

at $x = 1, \; y = -1, \; z = 1$

$$\nabla \cdot v = 2 - 6(-1)^2 \cdot 1 + 1$$

$$2 - 6 + 1 = 3$$

$$\nabla \cdot V = 3$$

The Curl of Vector

Let $E(x,y,z) = E_x\vec{i} + E_y\vec{j} + E_z\vec{k}$ be a continuous differentiable vector point function then curl of E is given by

$$\vec{i}\times\frac{\partial E_x}{\partial x} + \vec{j}\times\frac{\partial E_y}{\partial y} + \vec{k}\times\frac{\partial E_z}{\partial z} \text{ and}$$

is written as curl of E or $\nabla\times\vec{E}$

i.e

$$\nabla\times\vec{E} = \left(\vec{i}\times\frac{\partial}{\partial x} + \vec{j}\times\frac{\partial}{\partial y} + \vec{k}\times\frac{\partial}{\partial z}\right)\times$$

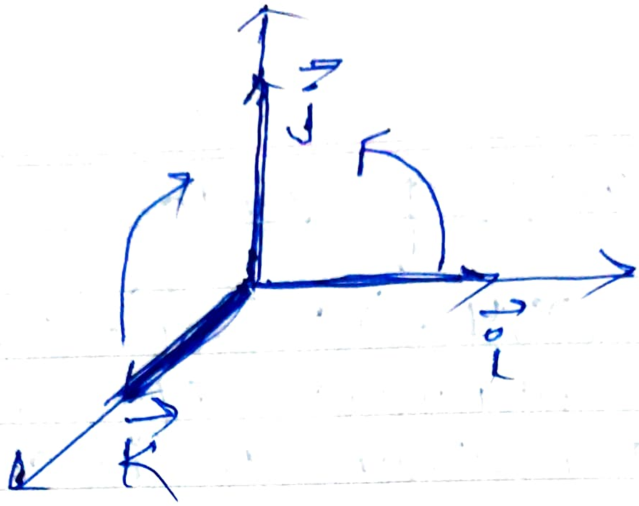
$$\left(E_x\vec{i} + E_y\vec{j} + E_z\vec{k}\right)$$

$$= (\vec{i}\times\vec{i})\frac{\partial E_x}{\partial x} + (\vec{i}\times\vec{j})\frac{\partial E_y}{\partial x} + (\vec{i}\times\vec{k})\frac{\partial E_z}{\partial x}$$

$$+ (\vec{j}\times\vec{i})\frac{\partial E_x}{\partial y} + (\vec{j}\times\vec{j})\frac{\partial E_y}{\partial y} + (\vec{j}\times\vec{k})\frac{\partial E_z}{\partial y}$$

$$+ (\vec{k}\times\vec{i})\frac{\partial E_x}{\partial z} + (\vec{k}\times\vec{j})\frac{\partial E_y}{\partial z} + (\vec{k}\times\vec{k})\frac{\partial E_z}{\partial z}$$

using cross product among unit vectors $\vec{i}$, $\vec{j}$ & $\vec{k}$ we get



$$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{K}\times\vec{K} = 0$$

$$\vec{i}\times\vec{j} = \vec{K}, \quad \vec{j}\times\vec{K} = \vec{i}, \quad \vec{K}\times\vec{i} = \vec{j}$$

$$\vec{i}\times\vec{K} = -\vec{j}, \quad \vec{K}\times\vec{j} = -\vec{i}$$

$$\vec{j}\times\vec{i} = -\vec{K}$$

$$\nabla\times\vec{E} = 0 + \vec{K}\frac{\partial E_y}{\partial x} - \vec{j}\frac{\partial E_z}{\partial x} - \vec{K}\frac{\partial E_x}{\partial y} + 0 + \vec{i}\frac{\partial E_z}{\partial y} + \vec{j}\frac{\partial E_x}{\partial z} - \vec{i}\frac{\partial E_y}{\partial z} + 0$$

$$= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\vec{i} + \vec{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \vec{K}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

which is in determinant form

$$\nabla \times \vec{E} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$\vec{A}\cdot(\vec{A}\times\vec{B}) = \begin{vmatrix} A_x & A_y & A_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$A_x[A_yB_z - A_zB_y] + A_y[A_zB_x - A_xB_z]$$

$$+ A_z[A_xB_y - A_yB_x]$$

$$A_x\cancel{A_y}B_z - A_x\cancel{A_z}B_y + A_y\cancel{A_z}B_x - A_y\cancel{A_x}B_z$$

$$+ A_z\cancel{A_x}B_y - A_z\cancel{A_y}B_x$$

- If the two row of the determinant are same, then the value of the determinant is zero

$$\therefore \vec{A}\cdot(\vec{A}\times\vec{B}) = 0$$

7219066236 garade vithal

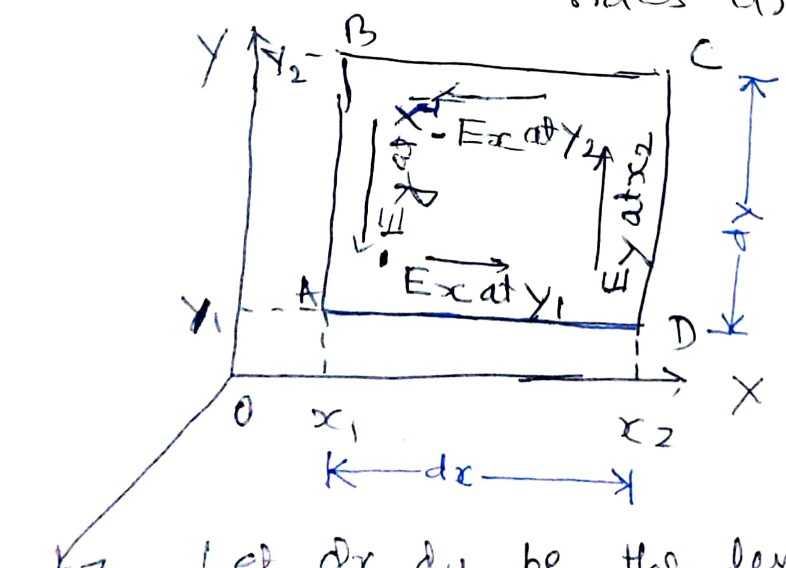
Physical defination of Curl of vector Field

Curl is the rate of change of vector field strength in a direction right angles to the field. Curl is the limiting value of circulation per unit area.

or in other words it is the line integral over a vector field round the path enclosing unit area.

Expression for the curl

To find the curl of a vector E interms of Cartesian co-ordinates consider very small rectangular area ABCD in the x-y plane with sides dx dy. Fig.



Let dx, dy be the lengths of the paths in the limiting case. Consider the the length dx at the bottom of the rectangale. It is in positive direction

The component of the field along AD is E_x at y_1 similarly the component of the field along DC is E_y at x_2 similarly along CB is $-E_x$ at y_2 and along BA is $-E_y$ at x_1. The last two terms are negative because on maintaining anticlockwise direction. The distance travelled along CB and BA is $-dx$ and $-dy$

$\therefore$ Total circulation

$$= [E_x \text{ at } y_1]dx + [E_y \text{ at } x_2]dy - [E_x \text{ at } y_2]dx - [E_y \text{ at } x_1]dy$$

$$= [(E_y \text{ at } x_2) - (E_y \text{ at } x_1)]dy - [(E_x \text{ at } y_2) - (E_x \text{ at } y_1)]dx$$

$$= [\text{increase in } E_y \text{ for a distance } dx]dy - [\text{increase in } E_x \text{ for a distance } dy]dx$$

$$= \frac{\partial E_y}{\partial x} dx dy - \frac{\partial E_x}{\partial y} dy dx$$

rate of change per unit length $\times$ total length total circulation X–Y plane

total $\rightarrow$ $= \left(\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} \right) dy\, dx$

But $dx\, dy =$ area

$\therefore$ Circulation or line integral of the path per unit area Area = unit area.

$$= \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}$$

xy

This is the component of the curl of the vector field in the z-direction

$$\therefore (\text{Curl } \vec{E})_z = \vec{K}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

Similarly by orientating the area parallel to yz and then parallel to zx plane we get

yz $\quad (\text{Curl } \vec{E})_x = \left(\dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z} \right) \vec{i}$

zx $\quad (\text{Curl } \vec{E})_y = \left(\dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x} \right) \vec{j}$

$\therefore$ Total curl $= (\text{Curl } \vec{E}_x + \text{Curl } \vec{E}_y + \text{Curl } \vec{E}_z)$

Circumscribe - draw a line round આવતાં વર્તુળ કાઢવો

whirling - turn round rapidly

periphery - out side the sphere

eddy — circular ~~mom~~ movement

$$= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\vec{i} + \vec{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)$$

$$+ \vec{K}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

putting above eq in the determinant form we get -

$$\text{Curl } \bar{E} = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$\text{since } \vec{\nabla} \times \vec{E} = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$\therefore$ Curl $\vec{E} = \vec{\nabla} \times \vec{E}$ Thus the cross product of the vector operator and a vector field E is called curl of E.

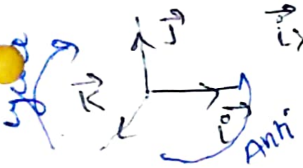
The curl of vector ~~field~~

Let $E(x,y,z) = E_x\vec{i} + E_y\vec{j} + E_z\vec{k}$ be a contineous differentiable vecte point function then the curl of E is given by

$$\vec{i}\times\frac{\partial E_x}{\partial x} + \vec{j}\times\frac{\partial E_y}{\partial y} + \vec{k}\frac{\partial E_z}{\partial z}$$

and is written as curl E or $\nabla\times E$

i.e. $$\nabla\times E = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\left(E_x\vec{i} + E_y\vec{j} + E_z\vec{k}\right)$$



$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = 0$

$i\times j = k$, $\vec{j}\times\vec{k} = \vec{i}$, $\vec{k}\times\vec{i} = \vec{j}$

$\cancel{\vec{k}\times\vec{i}} = -\vec{j}$, $\vec{k}\times\vec{j} = -\vec{i}$, $\vec{j}\times\vec{i} = -k$

$\vec{i}\times\vec{k} = -\vec{j}$ $\quad \vec{i}\times\vec{k}$

$$= (\vec{i}\times\vec{i})\frac{\partial E_x}{\partial x} + (\vec{i}\times\vec{j})\frac{\partial E_y}{\partial y} + (\vec{i}\times\vec{k})\frac{\partial E_z}{\partial z}$$
$$+ (\vec{j}\times\vec{i})\frac{\partial E_x}{\partial x} + (\vec{j}\times\vec{j})\frac{\partial E_y}{\partial y} + (\vec{j}\times\vec{k})\frac{\partial E_z}{\partial z}$$
$$+ (\vec{k}\times\vec{i})\frac{\partial E_x}{\partial x} + (\vec{k}\times\vec{j})\frac{\partial E_y}{\partial y} + (\vec{k}\times\vec{k})\frac{\partial E_z}{\partial z}$$

$$= 0 + \vec{k}\frac{\partial E_y}{\partial y} - \vec{j}\frac{\partial E_z}{\partial z} - \vec{k}\frac{\partial E_x}{\partial x} + 0 + \vec{i}\frac{\partial E_z}{\partial z}$$
$$+ \vec{j}\frac{\partial E_x}{\partial x} - \vec{i}\frac{\partial E_y}{\partial y} + 0$$

$$= \left(\frac{\partial E_z}{\partial z} - \frac{\partial E_y}{\partial y}\right)\vec{i} + \vec{j}\left(\frac{\partial E_x}{\partial x} - \frac{\partial E_z}{\partial z}\right) + \vec{k}\left(\frac{\partial E_y}{\partial y} - \frac{\partial E_x}{\partial x}\right)$$

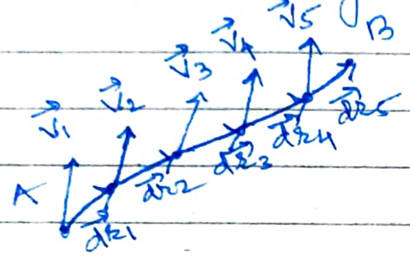
$$\nabla \times E = \begin{vmatrix} \vec{i} & \vec{j} & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

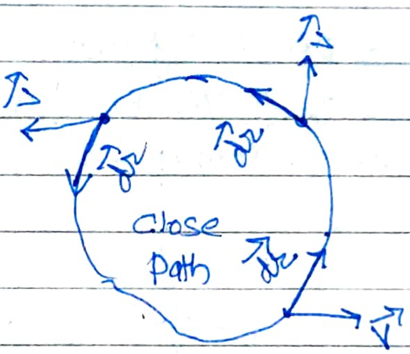
The vector integration

Most important integrals in the vector analysis are line integral, surface integral and volume integral.

i) The line integral.

The integration of a vector along a curve is called line integral.





Consider the line joining the points A and B. we can devide the line into small segments $d\vec{r_1}, d\vec{r_2}, d\vec{r_3}, d\vec{r_4} \cdot d\vec{r_5} \ldots$ having the values of vectors $\vec{V_1}, \vec{V_2}, \vec{V_3}, \vec{V_4}, \vec{V_5} \ldots$. Then $\vec{V} \cdot d\vec{r}$ is the scalar product of the length of the segment and the components $\vec{V}$ in its direction.

Thus

$$\vec{V} \cdot d\vec{r} = V \cos\theta \, dr$$

Then along line AB

$$\vec{V_1} \cdot d\vec{r_1} + \vec{V_2} \cdot d\vec{r_2} + \vec{V_3} \cdot d\vec{r_3} + \vec{V_4} \cdot d\vec{r_4} + \ldots = \sum \vec{V} \cdot d\vec{r}$$

if the length of the segment is very small then the summation may be replaced by integral.

$$\sum \vec{V} \cdot d\vec{r} = \int_A^B \vec{V} \cdot d\vec{r}$$

$$= \int_A^B V \cos\theta \, dr$$

is called line integral of vector field $\vec{V}$ along the line AB

If the path is closed and the point A and B coincide the line Integral is called the circulation of vector function $\vec{V}$ and is given by

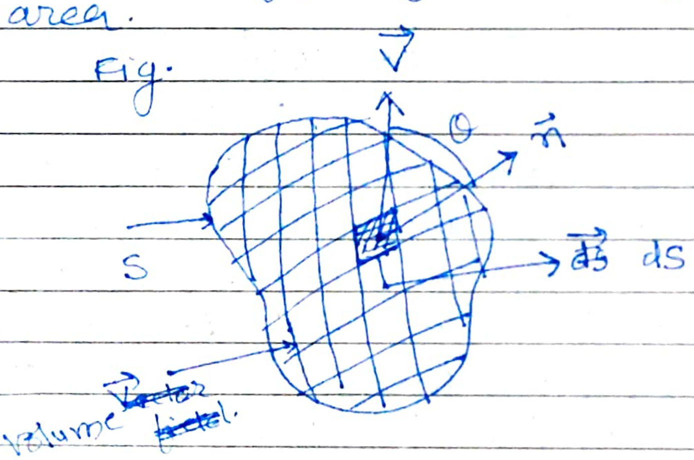
$$\text{circulation} = \oint \vec{V} \cdot \vec{ds}$$

where $\oint$ denotes integral round the closed path.

ii) The surface integral.
(Flux of vector field)

The rate of flow of mass (energy) per unit area.

Fig.



Let us consider the vector field $\vec{V}$ and draw the surface S which encloses the volume V. Let us now take an element of area ds on surface. The unit vector $\vec{n}$ is the outward normal to the surface element. If the surface is open the unit vector is in the same direction as the vector concerned.

Let θ be the angle between $\vec{n}$ and $\vec{V}$

The component of $\vec{V}$ perpendicular to area ds

$\vec{V} \cdot \vec{n}$

The flux of vector field $\vec{V}$ through the element having area ds is

$$= \vec{V} \cdot \vec{n}\, ds$$

Total flux out through the surface S we integrate above equation total flux through outward S = $\iint_{surface} \vec{V} \cdot \vec{n} \, ds$

$= \iint_S V \cos\theta \, ds$

This is flux of vector field $\vec{V}$ through the surface S is called the surface integral.

iii) Volume integral

consider a volume V in space and let $\vec{A}(x, y, z) = A_x \vec{i} + A_y \vec{j} + A_z \vec{k}$ be a vector function defined at each point in vol. V. The integral of $\vec{A}$ over the volume is called volume integral

$\therefore$ Volume integral $= \int_V \vec{A}(x, y\, z) \, dV$

if we substitute for $\vec{A}$ interms of its components then

Volume integral. $= \int_V (A_x \vec{i} + A_y \vec{j} + A_z \vec{k}) \, dV$

$dV = dx\,dy\,dz$

$= \vec{i} \int_V A_x \, dx\,dy\,dz + \vec{j} \int_V A_y \, dx\,dy\,dz + \vec{k} \int_V A_z \, dx\,dy\,dz$

It is cleare that volume of a vector is a vector.

The Volume Integral can be expressed as a triple integral

$$\int_V \vec{A}(x, y\, z) \, dV = \iiint_V \vec{A}(x, y, z) \, dx\,dy\,dz$$

If $\vec{V} = xz^3\vec{i} - 2x^2yz\vec{j} + 2yz^4\vec{K}$

find curl $\vec{V}$

$$\nabla\times\vec{V} = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{K}\frac{\partial}{\partial z}\right)\times\left(xz^3\vec{i}-2x^2yz\vec{j}+2yz^4\vec{K}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right] + \vec{j}\left[\frac{\partial}{\partial z}(xz^3) - \frac{\partial}{\partial x}(2yz^4)\right]$$

$$+\vec{K}\left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]$$

$$= \vec{i}\left[2z^4 + 2x^2y\right] + \vec{j}\left[3z^2x - 0\right]$$

$$+\vec{K}\left[-2\times2\cancel{x}yz - 0\right]$$

$$= \left[2z^4 + 2x^2y\right]\vec{i} + 3z^2x\vec{j} - 4xyz\vec{K}$$

Gauss Divergence Theorem

The amount of flux diverging from a given volume in a vector field is also equal to amount of flux diverging from the surface which enclosed in that volume.

This equality of surface and volume integrals in respect of incoming and outgoing fluxes is called Gauss divergence theorem. This theorem unable us to transform a volume integral into surface integral.

$$\oint_{surface} \vec{\nabla}\cdot\vec{n}\,ds = \int_{volume} div\,\vec{V}\,dv.$$

or

$$\oint_{S} \vec{A}\cdot\vec{n}\,ds = \int_{V} \vec{\nabla}\cdot\vec{A}\,dv$$

S is any closed surface and V is volume inside it.

Consider the volume integral in Gauss theorem.

$$\int_{V} \vec{\nabla}\cdot\vec{A}\,dv = \iiint_{V} \nabla\cdot\vec{A}\,dx\,dy\,dz \quad \text{---} (1)$$

$dv = dx\,dy\,dz$ is the small volume element that the center of which the value of vector is $\vec{V}$

using the expression

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{---} (2)$$

$$\int_V \vec{\nabla}\cdot\vec{A}\,dv = \iiint_V \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dx\,dy\,dz \quad \text{--- (3)}$$

split up integral into three term

$$= \iiint_V \frac{\partial A_x}{\partial x}.dx\,dy\,dz + \iiint_V \frac{\partial A_y}{\partial y} dy\,dx\,dz$$

$$+ \iiint_V \frac{\partial A_z}{\partial z}.dz\,dx\,dy \quad \text{--- (4)}$$

integrate first term on right hand side with respect to x we get

$$\iiint_{V\,x_1}^{x_2} \frac{\partial A_x}{\partial x}.dx\,dy\,dz$$

$$\iint \left[A_x(x_2 y z) - A_x(x_1 y z)\right] dy\,dz$$

similarly integrate second and third term on right hand side w.r to y and z we get

$$\iint \left[A_y(x, y_2, z) - A_y(x, y_1 z)\right] dx\,dz$$

and $$\iint \left[A_z(x, y, z_2) - A_z(x, y, z_1)\right] dx\,dy$$

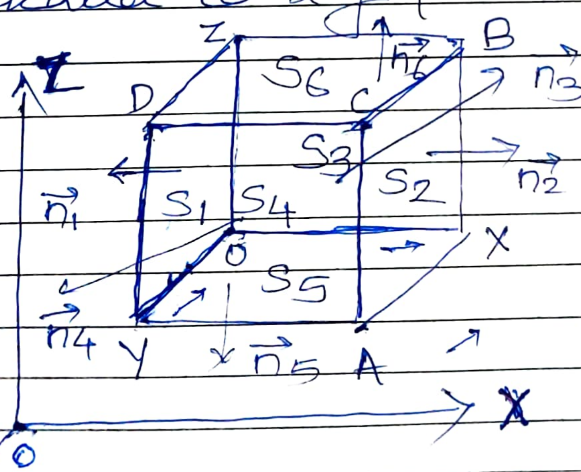
therefore equation (4) becomes

$$\int_V \vec{\nabla}\cdot\vec{A}\,dv = \iint \left[A_x(x_2,y,z) - A_x(x_1,y,z)\right] dy\,dz$$

$$+ \iint \left[A_y(x,y_2,z) - A_y(x,y_1,z)\right] dx\,dz$$

$$+ \iint \left[A_z(x,y,z_2) - A_z(x,y,z_1)\right] dx\,dy \quad \text{——(5)}$$

The first term on R.H.S. of above equation is integration over the surface S_1 & S_2 parallel to yz plane and second term is integration over the surface S_3 & S_4 parallel to xz plane and third term is integration over the surface S_5 & S_6 parallel to xy plane



$$\int_V \vec{\nabla}\cdot\vec{A}\,dv = \int_{S_2} A_x\,dy\,dz - \int_{S_1} A_x\,dy\,dz$$

$$+ \int_{S_4} A_y\,dx\,dz - \int_{S_3} A_y\,dx\,dz$$

$$+ \int_{S_6} A_z\,dx\,dy - \int_{S_5} A_z\,dx\,dy \quad \text{——(6)}$$

Now consider surface integral in Gauss theorem.

$$\oint_S \vec{A}\cdot\vec{n}\,ds \text{ over six surfaces.}$$

$$\oint_S \vec{A}\cdot\vec{n}\,ds = \int_{S_1} \vec{A}\cdot\vec{n}_1 ds_1 + \int_{S_2} \vec{A}\cdot\vec{n}_2 ds_2 + \int_{S_3} \vec{A}\cdot\vec{n}_3 ds_3$$

$$+ \int_{S_4} \vec{A}\cdot\vec{n}_4 ds_4 + \int_{S_5} \vec{A}\cdot\vec{n}_5 ds_5 + \int_{S_6} \vec{A}\cdot\vec{n}\, ds_6 \quad \text{--- (7)}$$

consider again

$$\int_{S_1} \vec{A}\cdot\vec{n}_1 ds_1 = YOZD \quad \text{--- (7a)}$$

the unit vector outward normal is $-\vec{i}$

$$\int_{S_2} \vec{A}\cdot\vec{n}_2 ds_2 = AXBC \quad \text{--- (7b)}$$

unit vector outward normal is $+\vec{i}$

$$\int_{S_3} \vec{A}\cdot\vec{n}_3 ds_3 = OXBZ \quad (7c)$$

unit ~~normal~~ vector outward normal is $-\vec{j}$

$$\int_{S_4} \vec{A}\cdot\vec{n}_3 ds_4 = YACD \quad \text{--- (7d)}$$

unit vector outward normal is $\rightarrow \vec{j}$

$$\int_{S_5} \vec{A}\cdot\vec{n}_5\, dS_5 = \overset{OXAY}{\cancel{OXAY}} \text{ --- } (7e)$$

unit vector outward normal is $-\vec{K}$

$$\int_{S_6} A\cdot\vec{n}_6\, dS_6 = DCBZ \text{ --- } (7f)$$

unit vector outward normal is $+\vec{K}$

- the area element over the different surface is

$$\left.\begin{aligned} dS_1 &= dS_2 = dy\,dz \\ dS_3 &= dS_4 = dx\,dz \\ dS_5 &= dS_6 = dx\,dy \end{aligned}\right\} \text{ --- } (8)$$

using this in equation (7) we may write

- $$\oint_S \vec{A}\cdot\vec{n}\, dS = \int_{S_1} \vec{A}\cdot(-\vec{i})\,dy\,dz + \int_{S_2} \vec{A}\cdot(\vec{i})\,dy\,dz$$

$$+ \int_{S_3} \vec{A}\cdot(-\vec{j})\,dx\,dz + \int_{S_4} \vec{A}\cdot(\vec{j})\,dx\,dz$$

$$+ \int_{S_5} \vec{A}\cdot(-\vec{K})\,dx\,dy + \int_{S_6} \vec{A}\cdot(\vec{K})\,dx\,dy$$

--- (9)

or

$$\oint_S \vec{A}\cdot\vec{n}\, dS =$$

Date : / /

$$= \int_{S_2} \vec{A}\cdot\vec{i}\, dy\,dz - \int_{S_1} \vec{A}\cdot\vec{i}\, dy\,dz$$

$$+ \int_{S_4} \vec{A}\cdot\vec{j}\, dx\,dz - \int_{S_3} \vec{A}\cdot\vec{j}\, dx\,dz$$

$$+ \int_{S_6} \vec{A}\cdot\vec{K}\, dx\,dy - \int_{S_5} \vec{A}\cdot\vec{K}\, dx\,dy$$

$$= \int_{S_2} A_x\, dy\,dz - \int_{S_1} A_x\, dy\,dz + \int_{S_4} A_y\, dx\,dz - \int_{S_3} A_y\, dx\,dz$$

$$+ \int_{S_6} A_z\, dx\,dy - \int_{S_5} A_z\, dx\,dy \quad \text{——— (10)}$$

unit vector magnitude 1

comparing equation (6) and (10) shows that two equation have same right hand side.

hence proved.

parallelepiped - 3D shape with six faces

parallelogram. - Four side fig opposite faces are parallel

STOKE'S THEOREM.

This theorem is used to transform line integral into surface integral and vice-versa.

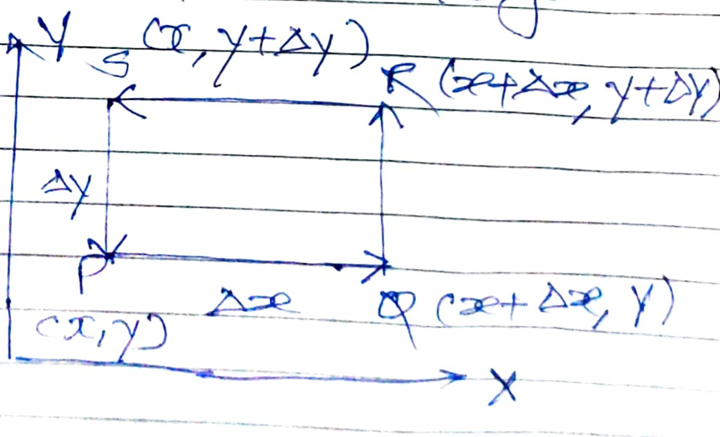
Statement:-

the line integral of vector $\vec{A}$ around any closed curve C is equal to the surface integral of normal component of the curl of vector $\vec{A}$ taken over any surface S of which the curve is a boundary edge.

Mathematically

$$\oint_C \vec{A} \cdot \vec{dl} = \iint_S (\nabla \times \vec{A}) \cdot \vec{dS}$$

Proof:- In order to find the connection between the line integral and curl of vector field, let us calculate the line integral of vector $\vec{A}$ around an infinitesimal rectangle of sides Δx and Δy lying in xy plane. Let x and y component of $\vec{A}$ at the point P be A_x and A_y respectively. We shall compute $\oint \vec{A} \cdot \vec{dl}$ around this rectangle.



The various contributions to the line integrals are as follows

The line integral along $PQ = A_x \Delta x$

The line integral along QR

$$= \left(A_y + \frac{\partial A_y}{\partial x}\Delta x\right) \times \Delta y$$

The line integral along $RS =$

$$-\left(A_x + \frac{\partial A_x}{\partial y}\Delta y\right) \times \Delta x$$

The line integral along $SP = -A_y \Delta y$

Adding all this we get

$$\oint_{PQRS} \vec{A}\cdot\vec{dl} = \cancel{A_x \Delta x} + \cancel{A_y \Delta y} + \frac{\partial A_y}{\partial x}\Delta x \Delta y$$

$$- \cancel{A_x \Delta x} - \frac{\partial A_x}{\partial y}\Delta y \Delta x - \cancel{A_y \Delta y}$$

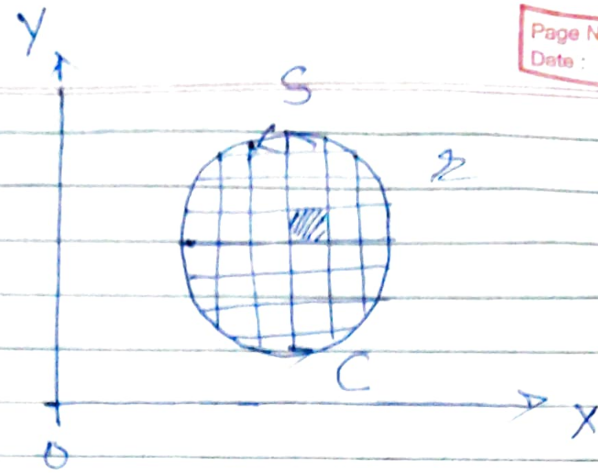
$$\oint_{PQRS} \vec{A}\cdot\vec{dl} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\Delta x \Delta y$$

$$\oint_{PQRS} \vec{A}\cdot\vec{dl} = (\nabla \times \vec{A})_z \, dS_{xy} \quad \text{—— (1)}$$

where $(\nabla\times\vec{A})_z$ is the magnitude of z component of curl $\vec{A}$.

$dS_{xy} = \Delta x \Delta y$ area of rectangle PQRS in xy plane.

consider a closed curve C in xy plane as shown in figure below



If we take the sum of line integral around meshes we obtained,

$$\sum_{r=1}^{\infty} \oint \vec{A}\cdot\vec{dl} = \sum_{r=1}^{\infty} (\nabla\times\vec{A})_z \, dS_{xy} \quad \text{---} (2)$$

Now the contributions to the line integrals of adjoining meshes neutralised each other because they are traversed in opposite directions. The only contributions which are not neutralized are those on the boundry c (or periphery) of the surfaces

Hence

$$\sum_{r=1}^{\infty} \oint \vec{A}\cdot\vec{dl} = \oint_c \vec{A}\cdot\vec{dl} \quad \text{---} (3)$$

where the line integral on right hand side of above equation is taken along the boundry curve c is positive sense.

The sum on the R.H.S of equation (2) reduces the following integral

$$\sum(\nabla\times\vec{A})_z ds_{xy} = \iint(\nabla\times\vec{A})_z dS_{xy}$$

putting this value in equation (2)
we get.

$$\oint_C \vec{A}\cdot\vec{dl} = \iint_S (\nabla\times A)_z dS_{xy}$$

Thus the line integral of vector $\vec{A}$ about the contour C of plane surface S is equal to the surface integral of normal component of curl $\vec{A}$ to the surface throughout the surface S.